KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2017/2018

FIRST SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN STATISTICS

SST 805: THEORY OF ESTIMATION

DATE: FRIDAY 2ND FEBRUARY 2018 **TIME: 5.30 P.M. – 8.30 P.M.**

INSTRUCTIONS: Answer question ONE and any other TWO questions.

Question One (30 Marks)

(a) Briefly explain the meaning of the following terms/phrases as used in statistical inference

(i) Estimate

(ii) Unbiasedness

(iii) Consistency

(iv) Uniformly Minimum Variance Unbiased Estimator [4 marks]

(b) Assume that all conditions of Crammer-Rao Inequality hold. Show that a necessary and sufficient condition for a estimator T to be most efficient is that it belongs to a 1-parameter exponential family. [9 marks]

(c) State without proof, the Rao-Blackwell Theorem and explain its significance in Statistical Inference Theory.

[4 marks]

(d) (i) State and prove the Rao-Blackwell-Lehmann-Scheffe Theorem.

(ii) Let $X_1, X_2, ..., X_n$ be iid $N(\theta, \sigma^2)$. Find the $UMVUE$ of θ^2 using the Rao-Blackwell-Lehmann-Scheffe Theorem.

[8 marks]

(e) Let $T_n = \delta(X_1, X_2, ..., X_n)$ be a Bayes Estimator having a constant risk. Show that T_n is a MiniMax Estimator of θ (*HINT: Consider the continuous case*). [5 marks]

Question Two (20 Marks)

(a) Prove the following results.

(i) If $X_1, X_2, ..., X_n$ are iid random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 \leq \infty$, then show that the sam mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is consistent.

(ii) If T_n is a sequence of consistent estimators of θ, and if b_n and c_n are two sequences of numbers such t $\underbrace{lim}_{n\to\infty} b_n = 1$ and $\underbrace{lim}_{n\to\infty} c_n = 0$ then the sequence of estimators $b_n T_n + b_n$is also consistent for θ.

[7 mar

(b) Let $X \sim N(0, \theta)$ and suppose that $T(x) = X^2$. Show that the statistic $T(x)$ is complete.

[7 mar

(c) Suppose $X_1, X_2, ..., X_n$ are independently distributed with common mean μ and variance σ_i^2, $i+1, 2, 3, ..., n$. Consid an estimator $T = \sum_{i=1}^{n} a_i X_i$. Derive a BLUE $T_{BLUE}$for μ when $\sigma_1^2 \neq \sigma_2^2 \neq ... \neq \sigma_n^2$ and hence conclude th $Var(T_{BLUE}) = \frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}$.

[6 mark

Question Three (20 Marks)

(a) Let d_1 and d_2 denote any two unbiased estimators of θ having known variances σ_1^2 and σ_2^2. Show that any estimato of the form $d = \lambda d_1 + (1 - \lambda)d_2$ will also be unbiased.

[4 marks]

(b) Suppose that T_n is an estimator which is asymptotically unbiased for a parameter $\theta \in \Theta$ such that $n \overset{lim}{\longrightarrow} \infty Var_\theta(T_n) =$ 0 for all $\theta \in \Theta$, then T_n is weakly consistent for θ.

[6 marks]

(c) Suppose that $X_1, X_2, X_3, ..., X_n$ are independent Bernoulli random variables, each having probability mass function given by $f(x/\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$ where θ is unknown. Further, suppose that θ is chosen from a uniform distribution on $(0, 1)$.Compute the Bayes estimator of θ.

[10 marks]

Question Four (20 Marks)

(a) Suppose $X_1, ..., X_n$ constitute a sample from a uniform distribution on $(0, \theta)$, where θ is unknown. Find the Maximum Likelihood Estimator of θ.

[9 marks]

(b) Let $X_1, ..., X_n$ be a random sample from X with density $f(x; \theta), \theta \in \Theta \subset R$. Confirm that the Maximum Likelihood Estimator $T_n = t(X_1, ..., X_n)$, satisfies, as $n \to \infty$:

(i) $T_n \xrightarrow{pr} \theta$

(ii) $n^{\frac{1}{2}}(T_n - \theta) \xrightarrow{pr} N(0; \frac{1}{I(\theta)})$, where $I(\theta)$ is the Fisher Information Number.

[11 marks]

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