KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF EDUCATION, BACHELOR OF SCIENCE AND BACHELOR OF ARTS

SMA 203: LINEAR ALGEBRA II

DATE: Thursday, 29th March, 2012 TIME: 11.00 a.m. – 1.00 p.m.

INSTRUCTIONS:

Answer question **ONE** and any other **TWO** questions.

Question One (30 marks)

a) Show whether or not the following are linear mappings:

i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x - y,\ y - z, 2x)$. (3 marks)

ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = xy$. (2 marks)

b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

$T(x, y, z) = (x + 2y, y - z, x + 2z)$. Find the dimension and a basis for the kernel of T. (4 marks)

c) Suppose $\lambda \neq 0$ is an eigenvalue of an invertible operator T. Show that λ^{-1} is an eigenvalue of T^{-1}. (4 marks)

d) Let T : $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T(x,y,z)=(2x-y-z,3x-2y+4z)$. Find the matrix of T relative to the basis $\{(1,1,1),(1,1,0),(1,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,2),(2,3)\}$ of $\mathbb{R}^2$. (4 marks)

e) Let T : $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $T(1,2)=(3,-1,5)$ *and* $T(0,1)=(2,1,-1)$. Determine $T(7,4)$. (4 marks)

f) Find the characteristic polynomial, minimal polynomial and eigenvalues of

$$A=\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}.$$ (6 marks)

g) Let $\{e_1,e_2\}$ be a basis of $\mathbb{R}^2$. If T : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear operator where by $T(e_1+e_2)=3e_1-2e_2$, $T(e_1-e_2)=7e_1+4e_2$. Find the matrix of T relative to the basis $\{e_1,e_2\}$. (3 marks)

Question Two (20 marks)

a) Find the value of x such that the determinant of $A=\begin{pmatrix} t+3 & -1 & 1 \\ 5 & t-3 & 1 \\ 6 & -6 & t+4 \end{pmatrix}$ *is zero.*

(5 marks)

b) Compute $\begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix}$ (3 marks)

c) Let T be the operator on the vector space V of 2 – square matrices over $\mathbb{R}$ defined by T(A) = MA, where $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find det T.

(Hint: Use the Basis $\left\{E_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

(3 marks)

d) Prove that similar matrices have the same determinant. (4 marks)

e) Similarity of matrices is an equivalence relation. Prove. (5 marks)

Question Three (20 marks)

a) Let T : $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by

$T(x,y,z) = (x+2y, y-z, x+2z)$. Find a basis and the dimension of

i) its image (3 marks)

ii) its kernel (3 marks)

b) Suppose $\{e_1, e_2\}$ is a basis of V and T : V $\to$ V is the linear operator for which $T(e_1) = 3e_1 - 2e_2$ *and* $T(e_2) = e_1 + 4e_2$. Suppose $\{f_1, f_2\}$ is a basis for V for which $f_1 = e_1 + e_2$ *and* $f_2 = 2e_1 + 3e_2$. Find the matrix of T in the basis $\{f_1, f_2\}$

(7 marks)

c) Let T : $\mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping and $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$ be the matrix of T with respect to the basis $\{v_1 = (1,0), v_2 = (2,3)\}$ of $\mathbb{R}^2$ and $\{w_1 = (1,0,0), w_2 = (1,1,0), w_3 = (1,1,1)\}$ of $\mathbb{R}^3$. Find T (v_1) and T (v_2) (7 marks)

Question Four (20 marks)

a) Let T : $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$T(x,y,z) = (2x+y, y-z, 2y+4z)$

i) Find all eigen values and a basis for each eigen space of T (5 marks)

ii) Determine whether or not T is diagonalizable.

(use the usual basis for $\mathbb{R}^3$. (2 marks)

b) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$. Is a similar to a diagonal matrix? If so, find one such matrix. (3 marks)

c) Consider a triangular matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$. Find the eigen values of A.

(3 marks)

d) Let λ be an eigen value of an operator $T:V \to V$. Let V_λ denote the set of all eigen vectors of T belonging to the eigen value λ. Show that V_λ is a subspace of V.

(5 marks)

e) Show that 0 is an eigen value of T if and only if T is singular. (2 marks)

Question Five (20 marks)

a) i) Let A be a square matrix. Differentiate between the characteristic and minimal polynomials of A. (2 marks)

ii) Find the characteristic and minimal polynomials of $A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$

(7 marks)

b) Show that the characteristic polynomial $D(x)$ and minimal polynomial $m(x)$ if a matrix A have the same irreducible factors. (3 marks)

c) State without proof the Cayley – Hamilton theorem and show its validity for the matrix. $A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$. (4 marks)

d) Let A be a square matrix with minimal polynomial $m(x) = x^2 - 5x + 6$. Prove that $A^3 = 19A - 30I$. (4 marks)