KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION (SCIENCE)

SMA 204: ALGEBRAIC STRUCTURES

DATE: MONDAY 26TH MARCH 2012 TIME: 2.00 P.M. – 4.00 P.M.

INSTRUCTIONS

Answer QUESTION ONE and ANY OTHER TWO questions.

Question 1

a) Suppose G is a group with the property that $x^2 = 1$ (the identity element) for all $x \in G$. Prove that G is abelian. (3 marks)

b) i) Express the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$ (2 marks)

as a product of disjoint cycles

ii) Let α and β denote the permutations of S_7 whose representations in disjoint cycle notation are

$$\alpha = (15)\ (27436)\ , \quad \beta = (1372)\ (46).$$

Express $\alpha\beta$ in disjoint cycle form. (2 marks)

c) List the elements in the group of symmetrics of a rectangle and construct the multiplication table for this group. (5 marks)

d) Let M denote the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where the entries are members of Z_7 and $a \neq 0$.

i) Show that M is a group with respect to matrix multiplication. (4 marks)

ii) Find the orders of elements

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$ (4 marks)

e) i) Define what is meant by a unit in a ring (2 marks)

ii) Find all the units in Z_6 (2 marks)

d) i) Define what is meant by an integral domain (2 marks)

ii) Define what is meant by a field (2 marks)

iii) Give an example of an integral domain which is not a field (2 marks)

Question 2

a) Draw the Cayley table for the set of complex numbers $\{1, \ -1, \ \ i, \ -i\}$ with respect to multiplication (4 marks)

b) i) Show that the inverse of ab is $b^{-1} \ a^{-1}$ (3 marks)

ii) Show that a cyclic group is abelian (4 marks)

c) Find the orders of the following permutations

i) $(1 \ 3 \ 6 \ 7 \ 2) \ (4 \ 5 \ 8)$ (2 marks)

ii) $(1 \ 3 \ 7 \ 2) \ (4 \ 6) \ (1 \ 5) \ (2 \ 7 \ 4 \ 3 \ 6)$ (3 marks)

d) List all the symmetries of a regular pentagon, regarded as permutations of vertices (corners) 1, 2, 3, 4, 5, labeled in cyclic order. (4 marks)

<u>Question 3</u>

a) Let α and β denote the functions defined (on a suitable subset of real numbers) by

$$\alpha(x) = \frac{1}{x} \quad , \quad \beta(x) = \frac{1}{1-x}$$

i) Find $(\alpha\ \beta)(x)$ (2 marks)

ii) Find $(\beta\ \alpha)\ (x)$

b) Define a binary operation $*$ on the set $\Re$ of real numbers by $x + y = xy + x + y$, where the operations on the right side of the equation have the usual meanings of multiplication and addition in $\Re$. Show whether or not $(\Re\ , *)$ is a group (3 marks)

c) i) State without proof the Lagrange's theorem (2 marks)

ii) Show that a group of prime order is cyclic. (3 marks)

d) i) Let $H = \{0\ ,\ 3\}$ be a subgroup of Z_6. List all the distinct left cosets of H in Z_6. (3 marks)

ii) Let $H = \{1, (13)\}$. List all the distinct left cosets of H in S_3 (3 marks)

iii) Show that if H is a subgroup of index 2 in a group G then the left coset gH is the same as the right Hg for all g in G. (2 marks)

Question 4

a) Let R be an integral domain such that $a^2 = a$ for all $a \in R$. Show that R is commutative. (4 marks)

b) Let u be the collection of all units in a ring $(R, + .)$.
Show that u is a group under multiplication. (4 marks)

c) i) Define what is meant by a subring of a ring. (2 marks)

ii) Let R be a ring and let a be a fixed element of R. Let $I_a = \{x \in R \mid ax = 0\}$. Show that I_a is a subring of R. (3 marks)

d) i) Define what is meant by a nilpotent element in a ring. (2 marks)

ii) List all the nilpotent elements in Z_6 (2 marks)

iii) Show that if F is a field, then F has no zero divisors. (3 marks)

Question 5

State whether the following are true or false

a) Every ring has a multiplicative identity. (2 marks)

b) Every ring with unity has at least two units. (2 marks)

c) Multiplication in a field is commutative (2 marks)

d) Every element in a ring has a multiplicative inverse. (2 marks)

e) The characteristic of nZ is n. (2 marks)

f) Every integral domain is a field

g) Every field is an integral domain (2 marks)

h) nZ has zero divisors if n is not a prime. (2 marks)

i) The nonzero elements of a field form a group under multiplication in the field. (2 marks)

j) Every ring with unity has at most two units. (2 marks)