KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2009/2010

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 301: REAL ANALYSIS II

DATE: FRIDAY 16TH APRIL 2010 **TIME: 11.00 A.M. – 1.00 A.M.**

INSTRUCTIONS

Answer Question ONE and any other TWO Questions.

QUESTION 1

a) Show that $f(x) = x^2 + 1$ is Riemann integrable on [1,2]. (4 marks)

b) For $n = 1, 2, 3, ----$, x real,

put $f_n(x) = \dfrac{x}{1+nx^2}$

Show that $\{f_n(x)\}$ converges uniformly to a function f, and the equations

$$f'(x) = \lim_{n \to \infty} f_n^1(x)$$

is correct if $x \neq 0$, but false if $x = 0$. (5 marks)

c) For what values of x does the series converge?

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}$$

(5 marks)

d) Find the n[th] partial sum of the series

$$\sum_{n=1}^{\infty} \frac{x}{[1+(n-1)x][1+nx]}$$ (5 marks)

[Hint: Resolve the n[th] term into partial fractions]

e) Let $f_n^{(x)} = nxe^{-nx^2}$, $n = 1,2,3,---,0 \le x \le 1$. Determine whether

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

and state whether or not $\{f_n(x)\}$ is uniformly convergent on [0, 1]. [6 marks]

f) Show that if the derivate $f^1(x)$ exists and is bounded on [a, b], then $f(x)$ is of bounded variation.

QUESTION 2

a) Show that the function f defined by

$$f(x) = \begin{cases} 0, & when\ x\ is\ rational \\ 1, & when\ x\ is\ irrational \end{cases}$$

is not Riemann integrable on any interval [a, b]. (7 marks)

b) Show that a function f is Riemann stieltjes integrable with respect to α on [a, b] if and only if for every $\varepsilon > 0$ there exists a partitions, P of [a, b] such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ (7 marks)

c) Let $f(x) = x$ and $\alpha(x) = x^3$. Find $L(P, f, \alpha)$ and $U(P, f, \alpha)$ corresponding to the partition

$P = \{1, 2, 3, 4, 5\}$ of $[1, 5]$. (6 marks)

QUESTION 3

a) i) Let $\{f_n(x)\}$, $n = 1,2,3,---$

be a sequence of functions defined on [a,b]. Define what is meant by $\{f_n(x)\}$ converges to $f(x)$ on $[a,b]$. (2 marks)

ii) Let $f_n(x) = \dfrac{x}{x+n}$ be defined on $[0\,,1]$. Find $\lim_{n\to\infty} f_n(x)$ on $[0\,,1]$.

(1 mark)

b) i) Define uniform convergence for a sequence of functions.

$\{f_n(x)\}$ on [a, b]. (2 marks)

ii) Consider the sequence $\{f_n(x)\}$, where

$f_n(x) = nx\,((1-x^2))^n \;,\; 0 \le x \le 1 \;,\; n = 1,2,3,---$

Show that

$$\lim_{n\to\infty}\left\{\int_0^1 f_n(x)dx\right\} \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

(6 marks)

c) i) Test for uniform convergence for the sequence $\{f_n(x)\}$, where

$f_n(x) = \dfrac{nx}{1+n^2x^2}$, for all real x. (5 marks)

ii) Is the sequence $\{f_n(x)\}$, $f_n(x) = \dfrac{1}{1+nx}$

uniformly convergent on [0,1] (4 marks)

QUESTION 4

a) For what values of x does the following series converge?

i) $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}$ (3 marks)

ii) $1 + 2x + 3x^2 + 4x^3 + ---$ (3 marks)

iii) $1+x^2 + x^4 + x^6 + ---$ (4 marks)

b) State and proof Weierstrass M-test (3 marks)

c) Show whether or not the following series of functions converge uniformly in the given sets.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in [a,b]$ (3 marks)

ii) $\sum \frac{\sin nx}{n^2}$, $x \in \Re$ (2 marks)

d) Prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ---$

[Hint: Use the fact that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + ---$ and integrate] (2 marks)

QUESTION 5

a) Define the term total variation of a function f on [a,b] and show that if the total variation is zero, then f is a constant on [a, b]. (6 marks)

b) Show that a bounded monotonic function is of bounded variation. (3 marks)

c) i) Find the variation of f(x) = sin x on $[0,\ 2\pi]$ with respect to the partition

$$P = \left\{0,\ \pi/2,\ \pi,\ \frac{3\pi}{z},\ 2\pi\right\}$$ (3 marks)

ii) Let $f(x) = \begin{cases} 1, & x \ \ rational \\ 0, & x \ \ irrational \end{cases}$

Is f of bounded variation on [0,1] ?

d) Show that a function of bounded variation is bounded. (5 marks)