KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

FIRST SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 302: GROUP THEORY

DATE: Wednesday 1st December, 2010 **TIME:** 2.00 p.m. – 4.00 p.m.

INSTRUCTIONS

Answer question One and any other TWO.

Question One - (30 marks)

a) (i) Express the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$

as a product of disjoint cycles. [2 marks]

(ii) Let $\alpha = (231)(45)$ and $\beta = (162)(34)$. Compute the product $\alpha\beta$ as a product of disjoint cycles. [3 marks]

b) Let $G = \mathbf{Z}_5 - \{0\}$ under multiplication.

(i) Construct the multiplication table for G. [2 marks]

(ii) Find the cyclic subgroups generated by 2 and 3. [2 marks]

(iii) Is G cyclic? [1 mark]

(iv) Find 3^{-1} and 4^{-1} [2 marks]

c) (i) Show that every subgroup of an abelian group is normal. [3 marks]

(ii) Let G be a group and let $H = \{x \in G \mid xg = gx, \forall\, g \in G\}$.

Show that H is a subgroup of G.

What is the name given to this subgroup? [3 marks]

d) Let $H = \{1, -1\}$ be a subgroup of the quaternion group G. Find all the left cosets of H in G. [4 marks]

e) (i) Define what is meant by a homomorphism from a group G to a group H. [1 mark]

(ii) Let **R** be the group of real numbers under addition and $\mathbf{R}^+$ be the group of positive real number under multiplication.

If f: $\mathbf{R} \rightarrow \mathbf{R}^+$ is defined by $f(x) = e^x$, $\forall x \in \mathbf{R}$, show that f is a homomorphism. [3 marks]

f) (i) Define what is meant by a simple group. [1 mark]

(ii) Give an example of two families of simple groups. [3 marks]

Question Two - (20 marks)

a) Let G be a group and H and K be subgroups of G. Show that $H \cap K$ is a subgroup of G. [3 marks]

b) (i) Define what is meant by a cyclic group. [2 marks]

(ii) Show that a group of prime order is cyclic. [4 marks]

c) (i) Define a normal subgroup H of a group G. [2 marks]

(ii) Let G be the set of all real 2 x 2 matrices $\begin{pmatrix} a & b \\ o & d \end{pmatrix}$ where $ad \neq o$ under multiplication. Let

$H = \left\{ \begin{pmatrix} a & b \\ o & 1 \end{pmatrix} \middle| a, b \in \mathbf{R} \right\}$ be a subgroup of G. Show that H is normal in G.

[4 marks]

d) (i) Let G be a group and $x, y \in G$. Explain the meaning of x is conjugate to y. [2 marks]

(ii) Find conjugacy class of $(23) \in S_3$. [3 marks]

Question Three – (20 marks)

a) (i) Define what is meant by a transposition. [2 marks]

(ii) State whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

is odd or even. [3 marks]

b) (i) Define what is meant by the order of an element of a group. [2 marks]

(ii) What is the order of the following permutations?

(1457)

(45) (237)

(14) (358) [3 marks]

c) (i) Let H be a sub-group of G. Define the normalizer, $N_G(H)$, of H in G. [2 marks]

(ii) Let H = {1, (12)}. Find the normalizer of H in S_3. Is H a normal subgroup of S_3? [3 marks]

d) (i) State without proof Langrange's theorem. [2 marks]

(ii) Show that if $a^2 = 1$ for all $a \in G$, then the group G is abelian. [3 marks]

Question Four – (20 marks)

a) Let H be a normal subgroup of G and let $f: G \to G/H$ be defined by $f(a) = aH$ for every $a \in G$. Show that f is a homomorphism. [2 marks]

b) Let $f: G \to H$ be a homomorphism of a group G into a group H. Show that Kerf is a normal subgroup of G. [4 marks]

c) Determine whether the following maps are homomorphisms. If a map is a homomorphism find its image and kernel;

(i) f: **Q*** $\to$ **Q***, where **Q*** is the multiplicative group of non zero rationals defined by $f(x) = |x|$. [4 marks]

(ii) f: $\mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+$ is the multiplicative group of positive real numbers and **R** is the additive group of real numbers defined by $f(x) = \log_{10} x$. [5 marks]

d) (i) Let G and $\overline{G}$ be two groups and $\phi: G \to \overline{G}$. Explain the meaning of ϕ is an isomorphism. [2 marks]

(ii) Let G be any group, g a fixed element in G. Define $\phi: G \to G$ by $\phi(x) = g\,x\,g^{-1}$. Prove that ϕ is an isomorphism onto G. [3 marks]

Question Five – (20 marks)

a) (i) Define a p-group. [2 marks]

(ii) Define a Sylow p-subgroup of a group G. [2 marks]

b) (i) State any of the three Sylow theorems. [3 marks]

(ii) What is the order of a Sylow 2-subgroup of a group of order 12? [1 mark]

(iii) What is the order of a Sylow 3-subgroup of a group of order 54? [2 marks]

(iv) List all the Sylow 2-subgroups of S_3. [3 marks]

c) Show that every group of order 45 has a normal subgroup of order 9. [7 marks]

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