KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 303: RING THEORY

DATE: MONDAY, 2ND APRIL 2012 **TIME: 8.00 A.M. – 10.00 A.M.**

INSTRUCTIONS: ANSWER QUESTION 1 AND ANY OTHER TWO QUESTIONS.

QUESTION 1

(a) (i) Define what is meant by a zero divisor in a ring. (2 marks)

(ii) Find all the zero divisors in $\mathbf{Z}_{12}$. (3 marks)

b) (i) Define what is meant by a unit in a ring. (2 mark)

(ii) Find all the units in $\mathbf{Z}_8$. (3 marks)

c) (i) Define what is meant by an ideal I of a Ring R. (2 marks)

(ii) Find all the ideals in $\mathbf{Z}_6$. (3 marks)

d) (i) Find all the cosets in the quotient ring $3\mathbf{Z}/12\mathbf{Z}$. (2 marks)

(ii) Show that if R is a ring with unity and I is an ideal of R containing a unit, then I = R. (3 marks)

e) (i) Define what is meant by a subset S is a subring of a ring R. (2 marks)

(ii) Give an example of a subring of a ring which is not an ideal. (3 marks)

f) (i) Let R and S be two rings and $f : R \rightarrow S$ be a mapping. Define what is meant by f is a ring homomorphism. (2 marks)

(ii) Let I be an ideal of a ring R. Show that the mapping $f : R \to R/I$ defined by $f(a) = a + I$ for all $a \in R$ is a ring homomorphism. (3 marks)

QUESTION 2

a) If $\mathbf{Z}_m$ is a ring of integers modulo m, then the set $M_2(\mathbf{Z}_m)$ of all 2 x 2 matrices over $\mathbf{Z}_m$ is a ring (you need not verify this).

(i) How many elements are there in $M_2(\mathbf{Z}_m)$? (2 marks)

(ii) Show the multiplication is not commutative in $M_2(\mathbf{Z}_2)$. (2 marks)

(iii) Which elements of $M_2(\mathbf{Z}_2)$ have a multiplicative inverse? (3 marks)

b) Let $f : R \to S$ be a ring homomorphism from a ring R into a ring S.

(i) Define what is meant by the kernel of f. (2 marks)

(ii) Show that ker f is an ideal of R. (2 marks)

(iii) Show that f is one-to-one if and only if ker $f = \{0\}$. (3 marks)

c) (i) Show that every field is an integral domain. (4 marks)

(ii) Give an example of an integral domain which is not a field. (2 marks)

QUESTION 3

(a) (i) Define what is meant by a maximal ideal. (2 marks)

(ii) Define what is meant by a prime ideal. (2 marks)

(b) (i) List all the maximal ideals of $\mathbf{Z}_{12}$. (3 marks)

(ii) List all the prime ideals of $\mathbf{Z}$. (3 marks)

(c) (i) Let R be a commutative ring with unity, and let $N \neq R$ be an ideal in R.

Show that R/N is an integral domain if and only if N is a prime ideal in R.

(4 marks)

(ii) Show that every maximal ideal in a commutative ring R with unity is a prime ideal. (2 marks)

(iii) Show that Z is a principal ideal domain. (4 marks)

QUESTION 4

$S = \{(a,b) | a,b \in \theta\}$, with operations $+$ and $\cdot$ forms a ring where the operations + and $\cdot$ are defined by

$(a,b)+(c,d)=(a+c,b+d)$

$(a,b)\cdot(c,d)=(ac+bc+ad,bd)$

(a) Show that S is a commutative ring with unity e. (6 marks)

(b) Find all the elements in *S* of the form $x \cdot x = e$. (6 marks)

(c) Determine the multiplicative inverse for (a,b) in S when it exists and the relationship between a and b when the inverse does not exist. (6 marks)

(d) Is S a field? Give reasons. (2 marks)

QUESTION 5

(a) Show that if F is a field, then every ideal in F[*x*] is principal. (5 marks)

(b) Compute the sum and product of the polynomials x^3+3x^2+2x+1 and x^2+4x+2 in $\mathbf{Z}_5[x]$. (5 marks)

(c) Find the quotient and remainder when $x^5+x^4+2x^3+x^2+4x+2$ is divided by x^2+2x+3. (5 marks)

(d) Find a gcd of x^3+x^2+x+1 and x^2+2 in $Z_3[x]$ and express the result in the form

$\lambda(x)(x^3+x^2+x+1)+\mu(x)(x^2+2)$

where $\lambda(x)$ and $\mu(x)$ are polynomials in $Z_3[x]$. (5 marks)

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