KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

FIRST SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF ARTS, BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION

SMA 402: FIELD THEORY

DATE: Tuesday 6th December, 2011 **TIME:** 11.00 a.m – 1.00 p.m

INSTRUCTIONS

Answer question ONE and other TWO questions.

Question 1

(a) (i) State what is meant by an element α of an extension E of a field f is algebraic over F. [2 marks]

(ii) Show that $\propto = \sqrt{\sqrt[3]{2} - i}$ is algebraic over **Q**. [4 marks]

(b) Find a basis of each of the following field extensions.

(i) $\mathbf{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$ over **Q** [2 marks]

(ii) $\mathbf{Q}\ \left(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24}\right)$ over **Q** [4 marks]

(c) (i) State what is meant by a root of a polynomial f(x)6 F[x]. [2 marks]

(ii) Let F be any field and f(x) = x2 + αx + $\beta \in$ F[x]

If E is any extension of F where F(x) has γ as a root, show that , which is in E is also a root of f(x). [4 marks]

(d) (i) Let E be an algebraic extension of a field F. State what is meant by two elements α, β ∈ E are conjugate over F. [2 marks]

(ii) Find all the conjugates of $\sqrt{1+\sqrt{2}}$ over **Q**. [4 marks]

(e) For each of the following polynomials in **Q** [x], find the degree over **Q** of the splitting filed over **Q** of the polynomial.

(i) $x^3 - 5$ [3 marks]

(ii) $(x^2 - 2)(x^3 - 1)$ [3 marks]

Question 2

(a) (i) State what is meant by the degree of a finite extension E over F. [1 mark]

(ii) Show that a finite extension E of a field F is an algebraic extension. [4 marks]

(b) Find the degree of each of the following field extensions.

(i) $\mathbf{Q}\left(\sqrt{2}, \sqrt{5}, \sqrt{10}\right)$ over **Q** [2 marks]

(ii) $\mathbf{Q}\left(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24}\right)$ over **Q** [2 marks]

(iii) $\mathbf{Q}\left(\sqrt{2}, \sqrt{6}+\sqrt{10}\right)$ over $\mathbf{Q}\left(\sqrt{3}+\sqrt{5}\right)$ [2 marks]

(c) (i) State what is meant by an irreducible polynomial over a field F. [1 mark]

(ii) Let $g(x) = x^4 + x^2 + 1 \in \mathbf{Q}[x]$. Find the roots of g(x). [3 marks]

(d) For each of the following algebraic numbers $\alpha \in \mathbf{C}$, find irr $(\alpha, \mathbf{Q})$ and deg $(\alpha, \mathbf{Q})$

(i) $\sqrt{2}+\mathrm{i}$ [3 marks]

(ii) $\sqrt{2}+\sqrt{3}$ [2 marks]

Question 3

(a) (i) Define what is meant by a splitting field of a polynomial f(x) over F[x]. [1 mark]

(ii) Find the splitting field of $x^4 - 1$ over **Q**. [4 marks]

(b) Find a basis of each of the following field extensions

(i) **Q**(i) over **Q** [2 marks]

(ii) $\mathbf{Q}\left(\sqrt{2}, \sqrt{6}\right)$ over $\mathbf{Q}\left(\sqrt{3}\right)$ [3 marks]

(c) Prove in detail that $\mathbf{Q}\left(\sqrt{3}+\sqrt{7}\right)=\mathbf{Q}\left(\sqrt{3}, \sqrt{7}\right)$ [5 marks]

(d) Find all conjugates of each of the following number over the given fields:

(i) $\sqrt{2}+\mathrm{i}$ over **Q** [2 marks]

(ii) $\sqrt{2}+\mathrm{i}$ over **R** [2 marks]

(iii) $\sqrt{1+\sqrt{2}}$ over $\mathbf{Q}\left(\sqrt{2}\right)$ [1 mark]

Question 4

(a) Let E be a finite extension of degree n over a finite F. Then if f has q elements, show that E has q^n elements. [5 marks]

(b) Show that a finite field E of p^n elements is the splitting field of $x^{p^n} - x$ over its prime subfield $\mathbf{Z}_p$. [5 marks]

(c) (i) Define an n^{th} root of unity and a primitive n^{th} root of unity in a field. [2 marks]

(ii) Let F be any field, and U_n be the set of all n^{th} roots of unity in F. Show that U_n is a group under field multiplication. [3 marks]

(d) (i) Find all generators of $(\mathbf{Z}_{7,}^{x}\, .)$ [2 marks]

(ii) Find all the 5^{th} roots of unity in $\mathbf{Z}_{11}$. [3 marks]

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