KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

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EXAMINATION FOR THE DEGREE OF BACHELOR OF EDUCATION

SMA 406: FUNCTIONAL ANALYSIS

DATE: SATURDAY, 9TH JULY 2011 **TIME: 8.00 A.M. – 10.00 A.M.**

INSTRUCTIONS: Answer question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a) Find the fixed points of the operator $T:\Re\to\Re$ defined by $Tx = x^3$. (3 marks)

b) Let $X = c$, where c is the space of all convergent sequences. Show that

$d(x,y) = \sup_{n\geq}\left|x_n - y_n\right|$ is a metric on c. (4 marks)

c) Prove that if $\left(P_n\right)_{n=1}^{\infty}$ is a sequence of paranorms on a linear space X, then

$p(x) = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{P_n(x)}{1+P_n(x)}$ is a paranorm on X. (5 marks)

d) Prove that if Y is a complete subset of a Banach space X, then Y is closed.

(3 marks)

c) Show that the functional $f: c[a,b]\to\Re$ define by

$f(x) = \int_a^b x(t)dt \quad , \quad x\in c[a,b]$ is linear and bounded (2 marks)

f) Show that if two norms $\|_n \cdot \|_1$ and $\| \cdot \|_2$ are equivalent and $\| x_n - x\|_1 \to 0$ as $n \to \infty$ then $\|x_n - x\|_2 \to 0$ as $n \to \infty$ for any sequence (x_n).

(2 marks)

g) Show that if T is a linear operator on a normed space X into a normed space Y, then T^{-1} is a linear operator on the range of T, provided T^{-1} exists.

(3 marks)

h. Prove that if $T : X \to Y$, where X and Y are normed spaces is continous at $x = 0$, then T is uniformly continous on X. (4 marks)

i) Show that strong convergence implies weak convergence with the same limit.

(3 marks)

QUESTION TWO (20 MARKS)

a) Define the following
   i) An Iteration (2 marks)
   ii) A Contraction map (2 marks)
   iii) A Complete metric space (2 marks)

b) Show that the subspace $S = (0,1]$ of $\Re$ is not complete. (3 marks)

c) Find the fixed points of the transformation T, where T is defined by

$$Tx = \frac{2x+5}{x-2}$$

(3 marks)

d) i) State Banach fixed point theorem. (2 marks)

ii) Let X be a complete metric space and $T : X \to X$ a contraction map. Prove that the Cauchy sequence $(x_n)^{\infty} n = 1$ of elements of X converge to the fixed point x in X and that x is unique. (6 marks)

QUESTION THREE (20 MARKS)

a) i) Define what is meant by a norm. (3 marks)

ii) Let $B(X,Y)$ be the family of all bounded linear operators from *X* to *Y*. Define $\|\cdot\|: B(X,Y) \to \Re$ by

$$\|T\| = \sup_{\|x\|=1} |Tx\|$$

Show that $\|T\|$ is a norm on $B(X,Y)$. (7 marks)

iii) Show that $\|x\| = \left| \lim_{n \to \infty} x_n \right|$ is not a norm on *c*, the space of all convergent sequences. (3 marks)

b) i) Define what is meant by equivalent norms. (2 marks)

ii) Prove that if *X* is a finite dimensional space and $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on *X* then $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$. (5 marks)

QUESTION FOUR (20 MARKS)

a) i) Define what is meant by a bounded linear operator. (2 marks)

ii) Let *X* and *Y* be normed linear spaces and $T: X \to Y$ a linear operator. Prove that *T* is continous on *X* if and only if *T* is bounded. (10 marks)

b) Define $T:(0,1) \to \Re$ by $Tx = 1/x$. Show that *T* is not uniformly continous on $(0,1)$. (4 marks)

c) Let $X = \left\{ x = (x_n) : \sum_{i=1}^{\infty} |x_i| < \infty,\ x_i \in \Re \right\}$.

Suppose X is a normed space with respect to the norm $\| x \| = \sup_i |x_i|$.

Show that $T_x = \sum_{i=1}^{\infty} x_i$ is a linear functional on X. (4 marks)

QUESTION FIVE (20 MARKS)

a) State precisely each of the following theorems;

i) Hahn-Banach theorem for normed spaces.

ii) Uniform boundedness principle.

iv) Open mapping theorem.

(2 marks each)

b) i) Distinguish between strong and weak convergence of a sequence (x_n) in a normed space X. (4 marks)

ii) Let (x_n) be a sequence in a normed space X. Prove that if $\lim X < \infty$, then weak convergence implies strong convergence. (6 marks)

c) If X is a compact metric space and $M \subset X$ is closed, show that M is compact.

(4 marks)

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