KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

OPEN, DISTANCE AND E-LEARNING

EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

<u>SMA 407: MEASURE AND INTEGRATION</u>

==

DATE: THURSDAY 7TH JULY 2011 TIME: 4.30 P.M. – 6.30 P.M.

<u>INSTRUCTIONS</u>

Answer question ONE and any other TWO.

<u>Question One (30 marks)</u>

a) Show that if $\mu^*(\mathrm{E})) = 0$, then E is measurable . (5 marks)

b) Let $A = \{x \in \Re : -4 \le x \le 5\} U \{x \in \Re : \quad 1 \le x \le 7\}$. Find $\mu^*(A)$. (2 marks)

c) If $f(x) = \frac{1}{2} + \cos x$, $0 \le x \le 2\pi$, find f^+ and f^- . (4 marks)

d) If f is a constant function, show that f measurable. (3 marks)

e) Prove that the empty set is measurable . (2 marks)

f) Let $f(x) = \begin{cases} \frac{1}{n} & , \ if \ |x| \leq n \\ 0 & , \ if \ |x| > n \end{cases}$

Calculate $\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)\,dx \ \ and \ \ \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx$ (4 marks)

g) Define what is meant by Boolean algebra. (2 marks)

h) Show that if a function f is integrable the $|f|$ is integrable. (3 marks)

i) Use a counter example to show that there exists non-measurable functions f with f^2 and $|f|$ measurable. (5 marks)

Question Two (20 marks)

a) i) Define what is meant by a measurable function. (2 marks)

ii) If f and g are measurable functions, show that fg is measurable. (4 marks)

b) For each extended real number α, show that the set $\{x \in X : f(x) = \alpha\}$ is measurable. (7 marks)

c) i) Define what is meant by a property p hold almost every where $a \cdot e$. (2 marks)

ii) If (X, x, μ) is a measure space and f, g, h are functions on X into $\Re e$ such that $f = g \ \ \mu - a \cdot e$ and $g = h \ \ \mu - a \cdot e$. Show that $f = h \ \ \mu - a \cdot e$. (5 marks)

Question Three (20 marks)

a) i) Define the outer measure of a set. (2 marks)

ii) Show that if $\mu^*(A) = 0$, then $\mu^*(AU\,B) = \mu*(B)$ (5 marks)

b) i) Define what is meant by a set function $T : x \to \Re e$ to be countably subadditive. (2 marks)

ii) Show that the Lebesgne outermeasure μ^* is countably subadditive. (5 marks)

c) Prove that for any set A and any finite collection of mutually disjoint measurable sets E_1, E_2, $--$, E_n, then

$$\mu^*\left(A \cap \{\bigcup_{k-1}^{n} E_k\} \right) = \sum_{k=1}^{n} \mu^* \; (A \cap E_k)$$ (6 marks)

Question Four (20 marks)

a) i) Define what is meant by a Lebesgue measurable set. (2 marks)

ii) Show that if E is Lebesgue measurable then E^c is also Lebesgue measurable. (4 marks)

b) Prove that if E_1 and E_2 are Lebesgue measurable, then $E_1\, U\, E_2$ is also Lebesgue measurable. (6 marks)

c) i) Define what is meant by a $\sigma-$algebra. (2 marks)

i) Let X_1 and X_2 be two $\sigma-$ algebras of subsets of the set X. show that $X_1 \cap X_2$ is also a $\sigma-$algebra. (6 marks)

Question Five (20 marks)

a) State the following theorems

i) Monotone Convergence Theorem (2 marks)

ii) Monotone Dominated Convergence Theorem (2 marks)

b) Let $f_n = \frac{1}{n} X_{[n,\infty]}$. Find $f = \lim_{n \to \infty} f_n$ and evaluate $\lim_{n \to \infty} \int f_n \ dx$

and $\int \lim_{n \to \infty} f_n \ dx$

Does the monotone convergence Theorem hold? Why ? (6 marks)

c) Find the Reiman and Lebesgue integrals of $\int_o^5 f(x) \ dx$, where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1; \\ 1, & \{1 \leq x < 2\} \ U \ \{3 \leq x < 4\} \\ 2, & \{2 \leq x < 3\} \ U \ \{4 \leq x \leq 5\} \end{cases}$$

(6 marks)

d) By use of the sequence $\{f_n\}$, $f_n = -\frac{1}{n} X_{[0,n]}$, show that the Fatus Lemma does not extend to functions that take negative values. (4 marks)