KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE, BACHELOR OF EDUCATION AND BACHELOR OF ARTS

SMA 407: MEASURE AND INTEGRATION

DATE: Friday 30th March, 2012 **TIME:** 8.00 a.m. – 10.00 a.m.

INSTRUCTIONS: *Answer question ONE and any other TWO questions.*

QUESTION ONE (30 MARKS)

a) (i) Define what is meant by a measure μ defined on a σ-algebra $\mathfrak{X}$. [2 marks]

(ii) Let X be any non-empty set and $\mathfrak{X}$ a σ-algebra of all subsets of X and let p be any fixed element of X. Let μ be defined for $E \in \mathfrak{X}$ by

$$\mu(E) = \begin{cases} 0, & \text{if } p \notin E \\ 1, & \text{if } p \in E \end{cases}$$

Show that μ is a measure on $\mathfrak{X}$. [5 marks]

b) Find the outer measure of the set

$$E = \cup \left\{ x \,\middle|\, \tfrac{1}{2^k} \le x < \tfrac{1}{2^{k-1}} \right\}$$ [4 marks]

c) (i) Define what is meant by a measurable set. [2 marks]

(ii) Prove that if E is measurable, then the complement E^c is also measurable. [3 marks]

d) Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is irrational number in } [-4, 4] \\ 0, & \text{if } x \text{ is rational number in } [-4, 4]. \end{cases}$

Find the Lebesgne integral of f on [-4, 4]. [4 marks]

e) Show that the constant function is measurable. [2 marks]

f) By use of the sequence $\{f_n\}$, where $f_n = -\frac{1}{n}\chi_{[o,n]}$, show that the Fatou's Lemma does not extend to functions that take negative values. [5 marks]

g) Show that if the function f is integrable, then |f| is integrable. [3 marks]

QUESTION TWO (20 MARKS)

a) (i) Define what is meant by a measurable function. [2 marks]

(ii) If f is a measurable function and c is a constant, show that cf is measurable. [4 marks]

(iii) For each extended real number α, show that the set $\{x \in X : f(x) = \alpha\}$ is measurable. [7 marks]

b) (i) Define what is meant by a property p to hold almost everywhere a.e. [2 marks]

(ii) If $(X, \mathfrak{X}, \mu)$ is a measure space and f, g, h are functions on X into **R**e such that f = g μ – a.e and g = h μ – a.e. Show that f = h μ – a.e. [5 marks]

QUESTION THREE (20 MARKS)

a) (i) Define what is meant by outer measure of a set E. [2 marks]

(ii) Show that if $\mu^*(E) = o$, then E is measurable. [5 marks]

(iii) Prove that for any set A and any finite collection of mutually disjoint measurable sets E_1, E_2, …, E, then $\mu^*\left[An\left(\bigcup_{i=1} E_i\right)\right] = \sum_{i=1}^{n} \mu^*\left(An\, E_i\right)$ [6 marks]

b) (i) Define what is meant by a set function $\tau : \mathfrak{X} \to$ **R**e to be countably subadditive. [2 marks]

(ii) Show that the Lebesgue outer measure μ^* is countably subadditive. [5 marks]

QUESTION FOUR (20 MARKS)

a) Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two σ-algebras of subsets of X. Show that $\mathfrak{X}_1 \cap \mathfrak{X}_2$ is also a σ-algebra. [5 marks]

b) (i) Show that the empty set ϕ is measurable. [2 marks]

(ii) Prove that if E_1 and E_2 are measurable, then $E_1 \cup E_2$ is also measurable.[6 marks]

c) (i) State the Monotone Convergence Theorem. [2 marks]

(ii) Let $f_n = \frac{1}{n}\chi_{[n,\infty]}\cdot$ Find $\lim\limits_{u\to\infty}\int f_n \, dx$ and $\int \lim\limits_{n\to\infty} f_n \, dx$. Does monotone convergence theorem hold? Why? [5 marks]

QUESTION FIVE – (20 MARKS)

a) Find the Riemann and Lebesgue integrals

$\int_0^5 f(x)\,dx$, where

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & \{1 \le x < 2\} \cup \{3 \le x < 4\} \\ 2, & \{2 \le x < 3\} \cup \{4 \le x \le 5\} \end{cases}$$ [6 marks]

b) (i) State the Lebesgue Dominated Convergence Theorem. [2 marks]

(ii) Let $f_n(x) = \begin{cases} \frac{x}{n^2}, & 0 < x < n \\ 0, & \text{otherwise} \end{cases}$

Evaluate $\lim_{n \to \infty} \int_0^n f_n(x)\,dx$. Does the Lebesgne Dominated Convergence Theorem hold?

Justify your answer. [5 marks]

c) If $f(x) = ½ + \cos x$, $0 \le x \le 2\pi$. Find f^+ and f^-. [4 marks]

d) Prove that if f is a non-negative extended real-valued function and if $E, F \in \mathfrak{X}$ with $E \subseteq F$, where $\mathfrak{X}$ is a σ-algebra of subsets of X, then

$\int_E f\,dx \le \int_F f\,dx$ [3 marks]

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