KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

INSTITUTIONAL BASED PROGRAMME (DECEMBER SESSION)

EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 500: MEASURE THEORY AND INTEGRATION

DATE: WEDNESDAY 27TH APRIL 2011 TIME: 9.00 A.M. - 12.00 NOON

INSTRUCTIONS
Answer ANY THREE questions.

QUESTION ONE (23 marks)

a) Let $f:\mathbf{R}\to\mathbf{R}$ be defined by $f(x)=\begin{cases}5 \text{ if } x=5''\\ 8 \text{ if } x=7''\\ 3 \text{ otherwise}\end{cases}$

Determine the upper and lower Riemann sums on [0.1, 50] using the partition 0.1<0.15<0.3<4<6<18<26<42<50. Is f Riemann integrable on [0.1, 50]? Explain. (8 marks)

b) *i)* Explain what is meant by a δ – algebra $\wp$ on X. (3 marks)

ii) Let X = $\{a,b,c,d,e\}$. Determine the δ – algebra generated by $A=\{a,d\}$ and $B=\{a,b,e\}$. (6 marks)

c) Discuss the difference between the Borel and extended Borel δ – algebras. Show that the Borel δ – algebra is contained in the extended Borel δ – algebra and that the two contain all singleton sets. (6 marks)

QUESTION TWO (23 marks)

a) Let P(**Z**) be the δ – algebra on **Z** consisting of all subsets of **Z,** the set of integers. Define

$f:\mathrm{P}(\mathbf{Z})\to\mathbf{R}$ by $\mu(E)=\begin{cases}4|E| & \text{if E is finite}\\ +\infty & \text{if E is infinite}\end{cases}$

Prove that μ is a measure. (6 marks)

b) Let μ be a measure defined on a δ – algebra $\wp$. If $\{E_n\}$ is a decreasing sequence in $\wp$ with $\mu(E_1)<+\infty$, prove that $\mu\left(\bigcap_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\mu(E_n)$ (7 marks)

c) Explain what is meant by **"a subset E of R is Lebesque measurable"** and prove that the class of Lebesque measurable sets is a δ – algebra on **R**. (10 marks)

QUESTION THREE (23 marks)

a) Let $f, g : \mathbf{R} \to \mathbf{R}$ where f is Lebesque measurable and $f = g$ m^* – almost everywhere, prove that g is Lebesque measurable. (7 marks)

b) Explain what is meant by a **simple function**, the **standard representation** and the **integral of a simple function.** (6 marks)

c) Let φ and ψ be simple functions on a measure space (X, $\wp$, μ), prove that

$$\int(\varphi+\psi)d\mu = \int\varphi d\mu + \int\psi d\mu .$$ (10 marks)

QUESTION FOUR (23 marks)

a) Differentiate between, **almost everywhere convergence, almost uniform convergence, L_p convergence** and **convergence in measure** (4 marks)

b) Prove that convergence in L_p implies convergence in measure (6 marks)

c) State Vitalli's Convergence Theorem (4 marks)

d) State and prove the Monotone Convergence Theorem (10 marks)

QUESTION FIVE (23 marks)

a) Let (X, $\wp$) and (Y, ζ) be δ – algebras,

i) Explain what is meant by a measurable function $f : \mathrm{X} \to \mathrm{Y}$ (2 marks)

ii) If (Y, ζ) = (**R**, **B**) where **B** is the Borel δ – algebras, show that $f : \mathrm{X} \to \mathbf{R}$ is measurable if $\mathrm{A}_\alpha = \{x \in X : f(x) > \alpha\} \in \wp$ for all $\alpha \in \mathbf{R}$. (5 marks)

b) Let (X, $\wp$) be a measurable space and $f : \mathrm{X} \to \mathbf{R}$ be a $\wp$-measurable function. Prove that the following statements are equivalent for $\alpha \in \mathbf{R}$.

i) $\mathrm{A}_\alpha = \{x \in X : f(x) > \alpha\} \in \wp$

ii) $\mathrm{B}_\alpha = \{x \in X : f(x) \leq \alpha\} \in \wp$

iii) $\mathrm{C}_\alpha = \{x \in X : f(x) \geq \alpha\} \in \wp$

iv) $\mathrm{D}_\alpha = \{x \in X : f(x) < \alpha\} \in \wp$ (10 marks)

c) Let f and g be measurable real valued functions defined on X. Prove that fg is measurable. (6 marks)

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