KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 502: FUNCTIONAL ANALYSIS 1

DATE: Monday 26th March, 2012 **TIME:** 9.00 a.m. – 12.00 p.m.

INSTRUCTIONS:

*Answer **Question One** and any other **TWO** questions.*

Question one (30 marks)

a) Let X be a normed linear space. Show that

(i) $\left| \|x\| - \|y\| \right| \leq \|x - y\| \; \forall x, y \in X$ [5 marks]

(ii) The mapping $x \rightarrow \|x\|$ is continuous (in the sense that if $x_n \rightarrow x$ then $\|x_n\| \rightarrow \|x\|$).

[3 marks]

(iii) Addition and scalar multiplication are jointly continuous i.e., if $x_n \rightarrow x$ and $y_n \rightarrow y$ then $x_n + y_n \rightarrow x + y$ and if $x_n \rightarrow x$ and $a_n \rightarrow a$ then $a_n x_n \rightarrow ax$ as $n \rightarrow \infty$ (a_n and a are real scalars. [4 marks]

b) (i) Let X be a linear space over **R**. Define a linear subspace of X. [2 marks]

(ii) Let M be a linear subspace of a normed linear space X and M denotes the closure of M w.r.t the norm determined metric on X. Prove that $\overline{M}$ is a linear subspace of X.

[6 marks]

c) Show that the following norms are equivalent:

$\|x\|_1 = \max \{|x_1|, |x_2|\}$ and

$\|x\|_2 = \sqrt{(|x_1|^2 + |x_2|^2)}$ on $\mathbf{R}^2$ [4 marks]

d) Let X and Y be normed spaces, $T \in B(X,Y)$ and $\{x_n\}$ a sequence in X. If $x_n \rightarrow x_o$, show that $Tx_n \rightarrow Tx_o$ [3 marks]

e) Let X and Y be normed spaces and $T: X \rightarrow Y$ be any map. Define each of the following:

(i) The graph of T, G(T) [1 mark]

(ii) A closed linear map T. [1 mark]

f) Show that all linear operators defined on a finite dimensional normed space X into an arbitrary normed space Y are bounded. [3 marks]

Question Two (20 marks)

a) State, without proof, the Hahn Banach theorem for a normed linear space. [3 marks]

b) Use the theorem in (a) to show that if every bounded linear functional on X vanishes at $x_o \in X$, then $x_o = 0$ [4 marks]

c) State one of the consequences of Hahn-Banach theorem and prove it. [7 marks]

d) Let $\{x_n\}$be a sequence in a normed linear space X. Prove that if $\{x_n\}$ converges strongly then it is uniformly bounded. [6 marks]

Question Three (20 marks)

a) Let X be a linear subspace over R and A be any subset of X.

(i) Define the term liner hull of A. [2 marks]

(ii) Prove that the closed linear hull of A is the closure of the linear hull of A. [6 marks]

b) Let X, Y be normed linear spaces and $T: X \rightarrow Y$ be a linear transformation. Prove that the following statements are equivalent:

(i) T is bounded [6 marks]

(ii) T is uniformly continuous [4 marks]

(iii) T is continuous at some point $x_o \in X$ [2 marks]

Question Four (20 marks)

a) State each of the following theorems.

(i) The open mapping theorem. [3 marks]

(ii) The closed graph theorem [3 marks]

b) Give one application of each of the above theorems. [4 marks]

c) Let B be a Banach space and T be a bounded linear map from B to B i.e $T \in L(B)$. Let λ be a complex number such that $\|T\| < |\lambda|$. Show that $T - \lambda I$ is an invertible operator and its inverse is

$$\left(\frac{-1}{\lambda}\right)\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^{n}$$ [10 marks]

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