KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

INSTITUTIONAL BASED PROGRAMME (IBP) – AUGUST SESSION

EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 503: FUNCTIONAL ANALYSIS II

DATE: Friday, 30th December, 2011 TIME: 9.00 a.m. – 12.00 p.m.

INSTRUCTIONS: Answer question **ONE** and any other **TWO** questions.

Q1. [Compulsory - 30 marks]

(a) Define what is meant by a compact operator T on a Hilbert space H. Hence, show that every compact oeprator T on H is bounded. **(4 marks)**

(b) State the parallelogram law on an inner product space H. **(2 marks)**

(c) Let l_p^n be a normed space of all n-tuples $x = (x_1, x_2, \cdots x_n)$ of real numbers equipped with the norm,

$$||x|| = (\sum_{k=1}^{n} |x_k|^p)^{1/p},\ 1 \leq p < \infty.$$

Show that l_p^n is not an inner product space for $p \neq 2$. **(2 marks)**

(d) Show that every unitary operator is normal but the converse is not necessarily true. **(4 marks)**

(e) Let $x_n \to x$ and $y_n \to y$ in a Hilbert space H. Show that $\lim\limits_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle$. **(2 marks)**

(f) Define what is meant by each of the following: **(2 marks each)**

i. Reflexive space X

ii. Uniformly convex Banach space X.

(g) Let H be a Hilbert space and y be a fixed vector of H. Prove that the functional $f(x) = < x, y > \forall x \in H$ is a continuous linear functional. **[3 marks]**

(h) Prove that if T is a normal operator then $Tx = \lambda x$ if and only if $T^*x = \bar{\lambda}x$. **[3 marks]**

(i) Let $T : X \to Y$ be bounded linear operator on a Hilbert space X into a Hilbert space Y and T^* its adjoint. Let $\mathcal{N}$ and R be the null and range spaces of T respectively. Prove each of the following relations: **[3 marks]**

i. $(R(T))^{\perp} = \mathcal{N}(T^*)$,

ii. $\overline{R(T)} = (\mathcal{N}(T^*))^{\perp}$.

(j) Suppose that $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$. Show that x_n converges to x strongly. **[3 marks]**

Q2. [Optional-20 marks]

(a) Let M be a closed convex subset of a Hilbert space H and $\delta = \inf ||x||$. Prove that there exists a unique $x \in M$ such that $||x|| = \delta$. **(10 marks)**

(b) Let M and N be closed subspaces of a Hilbert space H such that $M^{\perp}N$. Show that the subspace given b: $M + N = \{x + y \in H : x \in M,\, y \in N\}$ is also closed. **[10 marks]**

Q3. [Optional- 20 marks]

(a) State and prove the Riesz Representation theorem on a Hilbert space H. **[10 marks]**

(b) Let $T : X \to Y$ be bounded linear operator on a Hilbert space X into itself. Prove each of the following: **[3 marks each]**

i. $||T|| = ||T^*||$,

ii. $||T^*T|| = ||T||^2$.

(c) Let $T : X \to X$ be bounded linear operator on a Hilbert space X. Prove that T is self-adjoint if and only if $\langle Tx, x \rangle$ is real. **[4 marks]**

Q4. [Optional-20 marks]

(a) Let y and z be fixed elements of a Hilbert space H and define $T : H \to H$ by $Tx =< x, y > z$. Show that T is compact. **[5 marks]**

(b) Prove that a sequence $\{x_n\}$ in a Hilbert space H is weakly convergent to the limit $x \in H$ if and only if
$\lim\limits_{n\to\infty} < x_n, z >=< x, z >, \forall z \in H$. **[4 marks]**

(c) i. Prove that every Hilbert space is uniformly convex. **[4 marks]**

ii. Let X be a uniformly convex Banach space. Prove that for any $d > 0$, $\varepsilon > 0$ and arbitrary vectors $x, y \in E$ with $||x|| \leq d$, $||y|| \leq d$ and $||x - y|| > \varepsilon$, there exists a $\delta > 0$ such that $||\frac{1}{2}(x + y)|| \leq [1 - \delta(\frac{\varepsilon}{d})]d$. **[4 marks]**

iii. Show that the spaces l_∞ with the usual sup norm is not uniformly convex. **[3 marks]**