KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

INSTITUTIONAL BASED PROGRAMME (IBP) – AUGUST SESSION

EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 507: GROUP THEORY II

DATE: Thursday, 29^{th} December, 2011 TIME: 9.00 a.m. – 12.00 p.m.

INSTRUCTIONS:

Answer question **ONE** and any other **TWO** questions.

QUESTION 1

a) Define a Sylow p – subgroup of a group G. (2 marks)

b) Let G be the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a, b, \in, \mathbb{Z}_5$, a ≠ 0, and the multiplication is done (mod 5)

i) Find $|G|$ and the cyclic subgroup H generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (6 marks)

` ii) Prove that H in (i) is a Sylow 5 – subgroup of G and is normal. (4 marks)

iii) Find a Sylow 2 – subgroup of G and show that it is cyclic and determine how many Sylow 2 – subgroups there are in G. (4 marks)

c) Construct the finite field GF (2^3) (6 marks)

d) i) Define GL (n, q) and SL (n, q). (2 marks)

ii) Show that $|SL(n,q)| = \dfrac{|GL(n,q)|}{q-1}$ (2 marks)

e) i) Define $PSL(n,q)$ *and* $PGL(n,q)$ (2 marks)

ii) Find $|PSL(2,5)|$ and $|PGL(2,7)|$ (2 marks)

QUESTION 2

a) i) Define the centre of a group. (2 marks)

ii) Show that the centre of a group is normal. (2 marks)

b) Show that the order of the centre of $SL(n,q)$ is $(n,q-1)$, the gcd of n and q – 1. (5 marks)

c) Show that $SL(n,q)$ is normal in $GL(n,q)$. (3 marks)

d) Show that the order of $GL(n,q)$ is $(q^n-1)(q^n-q)---(q^n-q^{n-1})$. (6 marks)

e) List all the elements of $GL(2,2)$ (2 marks)

QUESTION 3

a) Define the following terms

i) p – group (2 marks)

ii) Maximal subgroups (2 marks)

iii) Frattini subgroup (2 marks)

b) i) Let G be a group. Define what is meant by the commutator G' of G. (2 marks)

ii) Show that the commutor subgroup of $SL(n,q)$ and $GL(n,q)$ is $SL(n,q)$ if $n \geq 3$. (4 marks)

c) i) Find all the maximal subgroups of the quaternion group and the alternating group A_4 (2 marks)

ii) Find the Frattini subgroup of the quaternion group and of the alternating group A_4. (2 marks)

d) Let $\Phi(G)$ denote the Frattini subgroup of G. Show that if $H \leq G$, then $H \leq .\Phi(G)$ if and only if there is no proper subgroup K of G such that $KH = G$. (4 marks)

QUESTION 4

a) i) Define what is meant by a primitive element of a field. (2 marks)

ii) Let α be a primitive element of $GF\left(2^2\right)$. List all the elements of $SL(2,4)$ generated by $\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$ (2 marks)

b) i) Define what is meant by a transvection. (2 marks)

ii) Show that $SL(n,q)$ is generated by transvections. (4 marks)

c) Show that any transvection in $GL(n,q)$ can be put in the form.

$$\begin{pmatrix} 1 & . & . & . & 1 \\ & . & & & . \\ & & . & & . \\ & & & . & . \\ & & & & 1 \end{pmatrix},$$

where all the entries are zero apart from the visible 1^s. (4 marks)

d) i) State what is meant by a subnormal subgroup H of a group G.

(2 marks)

ii) Give an example to show that a subnormal subgroup of G is not necessarily normal in G. (2 marks)

iii) Show that if G is nilpotent, then every subgroup of G is subnormal in G.

(3 marks)