KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2011/2012

SECOND SEMESTER EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 565: THEORY OF ESTIMATION

DATE: Friday, 30th March, 2012 TIME: 9.00 a.m. – 12.00 p.m.

INSTRUCTIONS: Answer question **ONE** and any other **TWO** questions.

QUESTION ONE: 30 MARKS

(a) Let $X_1, X_2, ..., X_n$ be a random sample from Bernoulli distribution with parameter p. Show that

$$\{\delta_n(\overline{X})\}_{n=1}^{\infty}; \quad \delta_n(\overline{X}) = \frac{T + \frac{\sqrt{n}}{2}}{n + \sqrt{n}}, \quad T = \sum_{i=1}^{n} X_i$$

is a mean squared error consistent sequence of estimators of p.

(b) Let $X_1, X_2, ..., X_n$ be a random sample from $N(\mu, \mu), \quad \mu > 0$.

(i) Show that $S = \sum_{i=1}^{n} X_i^2$ is a sufficient statistic for μ.

(ii) Show that the M.L.E of μ is

$$\hat{\mu} = \frac{-1 + \sqrt{1 + 4\frac{s}{n}}}{2}, \quad s = \sum_{i=1}^{n} x_i^2$$

(c) Let $X_1, X_2, ..., X_n$ be a random sample from density

$$g(x, \alpha, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x}, & x > 0, \alpha, \beta > 0 \\ 0, \text{otherwise} \end{cases}$$

where both α and β are unknown. Determine a pair of jointly sufficient and complete statistics for α and β

(d) Consider a random sample $X_1, X_2. ...X_n$ from a population whose distribution is $f(x, \theta)$. Show that

$$E\left[\frac{\partial}{\partial\theta} lnf(x,\theta)\right]^2 = -E\left[[\frac{\partial^2}{\partial\theta^2} lnf(x,\theta)\right]$$

stating any assumptions made.

(e) Let $X_1, X_2, .., X_n$ be a random sample with likelihood

$$f(x|\alpha) = \begin{cases} \frac{kx^{k-1}}{\alpha^k}, & 0 < x < \alpha, \alpha > 0 \\ 0, \text{otherwise} \end{cases}$$

where k is a positive integer; and α have the prior density

$$g(\alpha) = \begin{cases} \frac{\beta\delta^\beta}{\alpha^{\beta+1}}, & \delta < \alpha < \infty, \beta, \delta > 0 \\ 0, \text{otherwise} \end{cases}$$

with known β and δ. Find the posterior distribution of α and hence the Bayes' estimator of α

QUESTION TWO : 20 MARKS

(a) Let $X_1, X_2, ..., X_n$ be a random sample from density

$$f(x;\theta) = \begin{cases} \theta x^{-\theta-1}, & x \geq 1 \\ 0, \text{otherwise} \end{cases}$$

Show that $T = \frac{1}{n}\sum_{i=1}^{n} logX_i$ is an unbiased estimator for $\frac{1}{\theta}$.

(b) Let $X_1, X_2, ..., X_n$ be a random sample from density

$$f(x;\gamma) = \begin{cases} \frac{3x^2}{\gamma^3}, & 0 < x < \gamma \\ 0, \text{otherwise} \end{cases}$$

Let $\overline{X}$ be the sample mean and $Z_n = \max(X_1, ..., X_n)$

(i) Find a function of $\overline{X}$ that is an unbiased estimator for γ.

(ii) Find a function of Z_n that is an unbiased estimator for γ.

(iii) compare the variances of the estimators obtained in (i) and (ii)

QUESTION THREE : 20 MARKS

(a) Briefly explain how the Rao Blackwell and Lehmann Scheffe theorems can be used to obtain the the uniformly minimum variance unbiased estimator (UMVUE) of an unknown parameter θ or its function $\psi(\theta)$ based on arandom sample from a given density $f(x, \theta)$.

(b) Let $X_1, X_2, ..., X_n$ be a random sample from density

$$f(x;\theta) = \begin{cases} \theta(1-\theta)^x, & x = 0, 1, 2, ...0, 0 < \theta < 1 \\ 0, \text{otherwise} \end{cases}$$

Find the the uniformly minimum variance unbiased estimator (UMVUE) of $\psi(\theta) = \frac{1-\theta}{\theta}$.

(c) Let $X_1, X_2, .., X_n$ be a random sample from a normal distribution with mean 0 and standard deviation σ.

(i) Obtain the Cramer Rao lower bound for the variance of unbiased estimators of σ^2.

(ii) Does the variance of the uniformly minimum variance unbiased estimator (UMVUE) of σ^2 attain this bound.

QUESTION FOUR: 20 MARKS

(a) Let $Y_1, Y_2, ..., Y_n$ be a random sample from density

$$g(y, \alpha, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma\alpha} y^{\alpha-1} e^{-\beta y}, & y > 0, \alpha, \beta > 0 \\ 0, \text{otherwise} \end{cases}$$

Find the method of moments estimates of α and β.

(b) Let $X_1, X_2, ..., X_n$ be a random sample from density

$$f(x;\beta) = \begin{cases} \beta e^{-\beta x}, & x > 0, \beta > 0 \\ 0, \text{otherwise} \end{cases}$$

(i) Show that $T_n = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for β.

(ii) Determine the maximum likelihood estimator of β and hence construct a 98% confidence interval for β based on the asymptotic distribution of mle.

(iii) Show that $E[X_1|T] = \frac{T}{n}$.

(iv) Find the UMVUE of β.

QUESTION FIVE: 20 MARKS

Let $X_1, X_2, ..., X_n$ be a random sample from likelihood

$$f(x;\beta) = \begin{cases} \theta(1+x)^{-(\theta+1)}, & x > 0, \theta > 0 \\ 0, \text{otherwise} \end{cases}$$

Also let the prior density of θ be

$$f(\theta, \alpha, \beta) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{\alpha-1}e^{-\beta\theta} & \alpha > 0, \theta > 0, \beta > 0 \\ 0, \text{elsewhere} \end{cases}$$

(i) Find the posterior distribution of θ.

(ii) Find the Bayes estimator of θ with respect to the squared error loss function.

(iii) Find the maximum likelihood estimator for θ.

(iv) Show that the estimator obtained in part(ii) is a weighted average of the estimator in part(iii) and the prior mean.

(v) What is the value of the Bayes estimator as $n \to \infty$.

(vi) Is the Bayes estimator a minimax estimator?