KENYATTA UNIVERSITY

UNIVERSITY EXAMINATIONS 2010/2011

INSTITUTIONAL BASED PROGRAMME

DECEMBER 2010 SESSION EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 600: OPERATOR THEORY I

DATE: Friday 29th April, 2011 **TIME:** 9.00 a.m. – 12.00 p.m.

INSTRUCTIONS:

Answer question **ONE** and any other **TWO** Questions.

Q1. [Compulsory - 30 marks]

(a) Define a projection $P : H \to H$ on a Hilbert space H. **[2 marks]**

(b) If $\{P_n\}_{n=1}^{\infty}$ is a sequence of projections defined on a Hilbert space H and $P_n \to P$ as $n \to \infty$, then show that P is a projection on H. **[4 marks]**

(c) Let $P : H \to H$ be a projection operator on a Hilbert space H. Prove that the subspaces $\mathrm{Ran}(P)$, range of P and $\ker(P)$, kernel of P are orthogonal complements in H if and only if P is self adjoint operator. **[3 marks]**

(d) Let X, Y be Banach spaces and $B : X \to Y$ be a bounded linear operator such that $\| Bx \| \geq c \| x \| \; \forall x \in X$, where $c > 0$ is a constant. Show that

i. $\ker(B) = \{0\}$ and $\mathrm{Ran}(B)$ is closed. **[3 marks]**

ii. $\ker(B^n) = \{0\}$ and $\mathrm{Ran}(B^n)$ is closed for all $n \in \mathbb{N}$. **[2 marks]**

(e) Let X be a Banach space and $B : X \to X$ be a bounded linear operator. Show that all eigenvalues of B belong to the spectrum, $\sigma(B)$ of B. **[2 marks]**

(f) Consider the continuous linear operator $T : l_2 \to l_2$ on the Hilbert space l_2 given by $T(x_1, x_2, x_3, x_4, \cdots) = (x_1, -x_2, x_3, -x_4, \cdots)$, for any $x \in l_2$. Show that ± 1 are the eigenvalues of T. **[2 marks]**

(g) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H and let T^* be its adjoint.

i. Show that the spectrum of $T^*, \sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.[**2 marks**]

ii. Using the identity $(\lambda I - T)(\lambda I + T) = \lambda^2 I - T^2$, show that $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$. [**2 marks**]

(h) Show that if T is a positive operator then $(I+T)^{-1}$ exists. [**3 marks**]

(i) Let $T : H \to H$ be a bounded positive self-adjoint linear operator on H. Using the positive square root of T, show that

$$|< Tx, y >| \leq < Tx, x >^{\frac{1}{2}} < Ty, y >^{\frac{1}{2}}$$

for all $x, y \in H$. [**3 marks**]

(j) Prove that every linear operator T on a finite dimensional Hilbert space H is compact. [**2 marks**]

Q2. [Optional-20 marks]

(a) Let X be a vector space . Prove that if the operators $S, T : X \to X$ are such that $ST = TS$ and $TS : X \to X$ is invertible then both S and T are invertible. [**4 marks**]

(b) Let $B : X \to X$, be a bounded linear operator on Banach space X such that $||B|| < 1$. Prove that$(I - B)^{-1}$ exists,

$$(I - B)^{-1} = \sum_{n=0}^{\infty} B^n$$

and

$$\| (I - B)^{-1} \| \leq \frac{1}{1 - \| B \|}.$$ [**6 marks**]

(c) Let X be a Banach space and $A, B : X \to X$ be bounded linear maps. Let B be invertible and $\| A \| < \dfrac{1}{\| B^{-1} \|}$. Prove that$(A+B)^{-1}$ exists,

$$(A + B)^{-1} = (\sum_{n=0}^{\infty}(-B^{-1}A)^n)B^{-1}$$

and

$$\| (A + B)^{-1} \| \leq \frac{\| B^{-1} \|}{1 - \| A \| \| B^{-1} \|}.$$

[**7 marks**]

(d) Determine $(\lambda I - A)^{-1}$ if
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A^2 = I, \text{ the identity matrix and } |\lambda| < 1.$$ **[3 marks]**

Q3. [Optional-20 marks]

(a) Show that every projection $P : H \to H$ on a Hilbert space H satisfy $0 \leq P \leq I$, where I is the identity operator. **[2 marks]**

(b) Let $\{P_i\}_{i=1}^n$ be a finite set of projections on a Hilbert space H. If a sum $P_1 + P_2 + P_3 + \cdots + P_n$ is a projection on H, then show that $||P_1x||^2 + ||P_2x||^2 + \cdots + ||P_nx||^2 \leq ||x||^2$. **[4 marks]**

(c) Let H be a Hilbert space and $P_1, P_2 : H \to H$ be projections on H. Prove that

i. $P = P_1P_2$ is a projection on H if and only if $P_1P_2 = P_2P_1$. **[4 marks]**

ii. Two closed subspaces Y and V of H are orthogonal if and only if their corresponding projections P_Y and P_V satisfy $P_yP_V = 0$, where $P_Y(H) = Y$. **[4 marks]**

(d) Let $P : H \to H$ be a projection on a Hilbert space H. Prove that

i. $Q = I - P$ is a projection on H. **[2 marks]**

ii. $PQ = QP = 0$. **[1 marks]**

iii. Ran(P) = $\ker(Q)$ and $\ker(P)$ = Ran (Q). **[2 marks]**

Q4. [Optional - 20 marks]

(a) Let X be a Banach space and $T : X \to X$ be a compact operator on X. Prove that if $\lambda \neq 0$ is an eigenvalue of T then there exists a constant $c > 0$ such that

$$\| (T - \lambda I)x \| \geq c \| x \| \; \forall x \in X.$$

[6 marks]

(b) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Prove that if there exists a constant $k > 0$ such that $|\langle Tx, x \rangle| \geq k||x||^2, \forall x \in H$, then T is invertible and $||T^{-1}|| \leq \frac{1}{k}$. **[3 marks]**

(c) i. Define what is meant by the numerical range, $Num(T)$ of an operator $T : H \to H$ on a Hilbert space H. **[2 marks]**

ii. Prove that for any bounded linear operator $T : H \to H$ on a Hilbert space H, the spectrum of $T, \sigma(T) \subseteq \overline{Num(T)}$. **[5 marks]**

iii. Let $T : H \to H$ be a self adjoint bounded linear operator on a Hilbert space H and let

$$m = \inf_{||x||=1} \langle Tx, x \rangle, \ M = \sup_{||x||=1} \langle Tx, x \rangle.$$

Show that the spectrum of T, $\sigma(T) \subseteq [m, M]$. **[4 marks]**

Q5. [Optional- 20 marks]

(a) Let H be a Hilbert space and M a closed linear subspace of H. Let $P : H \to M$ be a projection and $y \notin M$.

i. Prove that there is a unique $u = Py \in M$ such that $||y - Py|| = \inf\{||y-x|| : x \in M\}$. **[8 marks]**

ii. Show that $\langle y-Py, x \rangle = 0, \forall x \in M$. **[6 marks]**

(b) Let T be bounded linear operators on a Banach space X and x_o an arbitrary element of X. Show that the equation

$$x = \alpha T x + x_o$$

has a unique solution given by

$$\sum_{n=0}^{\infty} \alpha^n T^n x_o,$$

provided $|\alpha| ||T|| < 1$. **[6 marks]**