KENYATTA UNIVERSITY
UNIVERSITY EXAMINATIONS 2010/2011
INSITUTIONAL BASED PROGRAMME (DECEMBER SESSION)

EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE

SMA 630: PARTIAL DIFFERENTIAL EQUATION I

DATE: **Friday 29th April 2011** **TIME:** **9.00a.m – 12.00p.m**

INSTRUCTIONS: Answer any three questions

Question One (23 marks)

a) i) Consider the heat equation $\mu_t - D\mu = 0$

Show that the solution of this equation, of the form

$u(x,t) = \frac{1}{t^{\alpha}} V\left(\frac{x}{t^{\beta}}\right)$, is given by $\mu(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ [9marks]

ii) The function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{2}} & x \in \Re^n \,,\ t > o \\ 0 & x \in \Re^n \,,\ t < o \end{cases}$$

is called the fundamental solution of the heat equation. For each $t > 0$, show that

$\int_{\Re^n} \Phi(x, t) dx = 1.$ [4marks]

Show that

i) $U \in C^{\infty}(\Re^n \ X\ (o, \infty)$

ii) $U_t\ (x,t) - \ DU\ (x,t) = o\ ,\ x \ \in \ \Re^n\ , t > o$

iii) $\lim u(x,t) = g(x^0)$ for each $x^o \in \Re^n$,

$(x,t) \rightarrow (x^o\ o)$ [4marks]

Question Two (23 marks)

a) Given that

$U_t\ -\ DU\ = f\ in\ U_T$

$U = g\ on\ \Gamma_T$

Where $\cup_T = \ B^o\ (y, \gamma)$ is a sphere centered at y with radius γ and Γ_T is the boundary of the sphere $B^o(y\ ,\ \gamma)$. Show by use of energy methods that there exists atmost one solution $U \in C_1^2(\bar{U}_T)$ of the above. [10marks]

b) Given the wave equation

$U_{tt} = Du = 0\ in\ \Re^n\ x\ \{o, \infty\}$

$U = g\ on\ \Re^n\ x\ \{t = o\}$

$U_t\ = h\ \ on\ \Re^n\ x\ \{t = o\}$

Fix $x \in \Re^n$ and let u be the solution of the above initial value problem. Define

$U(x', y, t) = f\ u\ (y,t) ds(y)$

$\partial B\ (x, \gamma)$

$G(x; \gamma) = \ f\ g(y) d\ s(y)$

$\partial B\ (x, \gamma)$

$H(x; \gamma) = \ f\ \ h\ (y) d\ s(y)$

$\partial B\ (x, y)$

Prove that

i) $U \in C^M(\Re_{\mp}^{-} x (o, \infty)$ [4marks]

ii) $U_{tt} - U_{rr} = \frac{n-1}{r} U_r = 0 \; in \; \Re_+ \; x \; (8, \infty)$

$U = \;\; G, U_t$ = H on $\Re_t x\{t = 03\}$ [9marks]

Question Three (23 marks)

a) Prove the multinomial theorem

$$(x_1 + x_2 + .. + x_n)^K = \sum_{|\propto|} (|\propto|) \; x^{\propto}$$ where

$\binom{|\propto|}{\propto} := \frac{|\propto|!}{\propto!}, \propto !, = \propto_1 ! \;\; \propto_2 !, \;\; \propto_3 \; ! ... \propto_n \;\; !$ and $x^{\propto} = x_1^{\propto_1} ... x_n^{\propto_n}$, the sum is taken over multi-indices $\propto = (\propto_1 ..., \propto_n) with \; |\propto| = k,$ [6marks]

b) Prove Leibniz' formula

$$D^{\alpha(}(uv) = \sum_{\beta < \in} \binom{\alpha}{\beta} D^{\beta} u \, D^{\alpha-\beta} V,$$

Where $U, v \; i \; \Re^n \to \Re \; are \; smooth, \binom{\propto}{\beta} = \frac{\propto!}{\beta!(\propto-\beta)!}$ and $\beta \;\; \leq \propto means \; \beta_l \;\; \leq \propto_l \;\;\; (i = 1, ..., n),$

[6marks]

c) Assume that $f : \Re^n \to \Re \; is \; smooth, \;\; Prove \; that$

$$f(x) = \sum_{\|\alpha\|} \geq k \frac{1}{\alpha!} D^{\alpha} \; f(o) x^{\alpha} + 0\left([\alpha]^{k+1}\right) as \; x \to o$$ for each $k = 1, 2 ...$ [6marks]

d) State three physical interpretation of Laplace's equation and name their corresponding laws. [5marks]

Question Four (23 marks)

a) Prove that the Laplace's equation $Du = o. where \; u: \mathcal{R}^n \to \Re, is \; rotation.$

[6marks]

b) Let $x \in \;\; \Re^n$ and $\gamma = |x| = (x_1^2 + x_2^2 + .. + x_n^2)^{\frac{1}{2}}$ by taking $u(x) = v(\gamma),$

Show that $D \; u = o$ can be expressed as $v''(\gamma) + v'\left(\frac{n-1}{\gamma}\right) = o$ [8marks]

c) Show that $D\,u = o$ can be expressed as $v''(\gamma)+v'\left(\frac{n-1}{\gamma}\right)=o$ in (b) is given by

$$v(\gamma)=\begin{cases} b\ \log\gamma\ \ for\ n=2 \\ \dfrac{b}{\partial^{n-2}}\ \ for\ n\geq 3. \end{cases}$$

Where b is a constant. [9marks]

Question Five (23marks)

a) Suppose that $f\ \in\ C_c^2((\Re^u)$ and

$$u(x) = \int_{\Re^n} \Phi\ (x-y)f\ (y)dy = \begin{cases} -\frac{1}{2\pi}\int_{\Re^2} \log(1x-\frac{y}{} f(y)dy\ n=2 \\ \frac{1}{n(-2)} \propto (n)\ \int_{\Re^n} \frac{f(y)}{1x-y/^{n-2}}\ \ n\ 2{,}3 \end{cases}$$

Where $\alpha(n)=\dfrac{\pi^{\frac{n}{2}}}{)(1+\frac{n}{2})}$

Prove that

i) $U\ \in\ C^2\ (\Re^n)$ [8marks]

ii) $-Du = f\ in\ \Re^n$ [10marks]